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# Multiplicative noise induces zero critical frequency

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Stochastic Bloch equations which model the fluorescence of two-level molecules and atoms, NMR experiments, and Josephson junctions are investigated to illustrate the profound effect of multiplicative noise on the critical frequency of a dynamical system. Using exact solutions and the cumulant expansion we find two main effects: (i) even very weak noise may double or triple the number of critical frequencies, which is related to an instability of the system, and (ii) strong multiplicative noise may induce a nontrivial zero critical frequency thus wiping out the overdamped phase.

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Many dissipative deterministic dynamical systems exhibit two phases of motion: an underdamped oscillatory behavior or an overdamped nonoscillatory motion. The transition between these common behaviors defines the critical frequency of the system, e.g., the critical frequency of the damped harmonic oscillator. Multiplicative noise is known to influence deterministic systems, in profound and surprising ways [1–11]. Here we show how a stochastic perturbation induces a zero critical frequency for a particular nontrivial choice of noise strength, thus completely wiping out the overdamped phase. This might be counterintuitive at first glance since we expect noise to work against oscillations; however as we soon demonstrate, in some cases the opposite situation is found. The second interesting result we obtain is that even weak multiplicative noise may induce a doubling or a tripling of the number of critical frequencies of a system (in a way defined later) a result which is related to an instability of the noiseless dynamical system. Our results show how multiplicative noise may influence the critical frequency of a system in profound ways.

We investigate the dynamics of the stochastic Bloch equation. The Bloch equation finds its applications, in many fields of physics ranging from nuclear magnetic resonance (NMR) [12,13] to single molecule spectroscopy [14,15] and Josephson's junctions [16]. We use the example of the optical Bloch equation, however, with minor modifications we may consider other systems, e.g., magnetic systems. In particular we consider a two-level electronic transition of an atom or a molecule interacting with a continuous wave laser and a stochastic bath. The optical Bloch equation for $\vec{Z}(t)=(u,v,w,y)$, where $(u,v,w)$ describes the usual Bloch vector, is

$$(d/dt)\vec{Z}(t)=M(t)\vec{Z}(t),$$

$$M(t)=\begin{pmatrix} -\frac{\Gamma}{2} & \delta_L(t) & 0 & 0 \\ -\delta_L(t) & -\frac{\Gamma}{2} & -\Omega & 0 \\ 0 & \Omega & -\Gamma & -\Gamma \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{1}$$

The initial condition is $\vec{Z}(0)=(0,0,-1/2,1/2)$ describing a system in the ground state and $y=1/2$ for all times. Here $\Gamma$ is the radiate emission rate and $\Omega$ is the Rabi frequency describing the interaction of the transition dipole of the system with the laser field (within rotating wave approximation). The stochastic detuning $\delta_L(t)=\omega_L-[\omega_0+\nu(t)]$ describes the interaction of the system with the bath in the spirit of the Kubo-Anderson line-shape theory [14,17,18], namely, $\delta_L(t)$ is a stochastic process describing spectral diffusion. $\omega_L$ is the laser frequency and $\omega_0$ is the absorption frequency of the two-level system. Spectral diffusion is found in many molecular, atomic, and magnetic systems and is well investigated [14,19]. The noise is called multiplicative since in Eq. (1) the spectral diffusion process multiplies the vector $\vec{Z}$. Beyond spectral diffusion the equations describe a two-level system in the process of resonance fluorescence, where the laser frequency exhibits fluctuations, namely, the detuning is a random function of time.

For the noiseless case, $\nu(t)=0$, with zero laser detuning, $\omega_L=\omega_0$, we have a simple damped harmonic oscillator for $w$ [20]

$$\ddot{w}+\left(\frac{3\Gamma}{2}\right)\dot{w}+\left(\Omega^2+\frac{\Gamma^2}{2}\right)w+\left(\frac{\Gamma^2}{4}\right)=0. \tag{2}$$

When $\Omega$ is larger than the critical frequency $\Omega_c=\Gamma/4$ the system exhibits underdamped Rabi oscillations, while when $\Omega<\Omega_c$ it decays to the steady state monotonically. The critical frequency provides the quickest approach of the amplitude of the damped harmonic oscillator to zero.

Now we consider the system in the presence of the spectral noise and investigate the average behavior $\langle w\rangle$. What will happen to the critical frequency of the system and can we choose parameters of the noise in such a way that the critical frequency of the noisy system is zero?

The formal solution of the problem is given in terms of the time-ordered exponential

$$\langle\vec{Z}(t)\rangle=\left\langle \hat{T}\left\{\exp\left(\int_0^t M(\tau)d\tau\right)\right\}\right\rangle\vec{Z}(0). \tag{3}$$

In practice it is generally difficult to find explicit solutions due to the combination of the time-ordering operator $\hat{T}$ and the average over the multiplicative stochastic process denoted with $\langle\cdots\rangle$ in Eq. (3). Here we find an exact solution for a dichotomic two state Kubo-Anderson process [18]. With this solution we will explore whether the motion is overdamped or underdamped. We later show that our findings are general beyond the exactly solvable two-state process. In particular we consider $\nu(t)=\nu h(t)$, where $h(t)=+1$ or $h(t)=-1$ describes the stochastic two state process with a rate $R$ for transitions between +1 and −1. Such a model is applicable in single molecule spectroscopy in glasses [15,21,22] and was used extensively for the line-shape theory of Kubo and Anderson.

We use Burshtein's method [23] of marginal averages to solve the Kubo-Anderson process with zero laser detuning $\omega_L=\omega_0$. We define the marginal average vector $[\langle\vec{Z}(t)\rangle_+,\langle\vec{Z}(t)\rangle_-]$ which is the average of $\vec{Z}(t)$ given that at time $t$ the stochastic process had the value $\pm 1$ correspondingly. The equation of motion for the marginal average vector is

$$\frac{\partial}{\partial t}\begin{pmatrix}\langle\vec{Z}\rangle_+\\ \langle\vec{Z}\rangle_-\end{pmatrix}=\begin{pmatrix}A_0^+-RI & RI\\ RI & A_0^- - RI\end{pmatrix}\begin{pmatrix}\langle\vec{Z}\rangle_+\\ \langle\vec{Z}\rangle_-\end{pmatrix}, \tag{4}$$

$$A_0^{\pm}=\begin{pmatrix}-\frac{\Gamma}{2} & \pm\nu & 0 & 0\\ \mp\nu & -\frac{\Gamma}{2} & -\Omega & 0\\ 0 & \Omega & -\Gamma & -\Gamma\\ 0 & 0 & 0 & 0\end{pmatrix}, \tag{5}$$

where $I$ is the identity matrix. The operators $A_0^+$ and $A_0^-$ are Bloch matrices corresponding to the state of the spectral diffusion $\delta_L=+\nu$ or $\delta_L=-\nu$, respectively. To solve the problem we must diagonalize the $8\times 8$ matrix in Eq. (4), then complex eigenvalues yield underdamped oscillatory modes while real eigenvalues correspond to overdamped modes. The eigenvalues $\{\lambda\}$ are found using the characteristic polynomial of Eq. (4)

$$\lambda(\lambda+2R)P_1(\lambda)P_2(\lambda)=0, \tag{6}$$

where two cubic polynomials are defined as

$$P_1(\lambda)=[8\Omega^2+4(\Gamma+\lambda)(\Gamma+2\lambda)]R+2(\Gamma+2\lambda)\Omega^2+(\Gamma+\lambda)\times[(\Gamma+2\lambda)^2+4\nu^2],$$

$$P_2(\lambda)=8(\Gamma+2\lambda)R^2+2[4\nu^2+(\Gamma+2\lambda)(3\Gamma+4\lambda)]R+2(\Gamma+2\lambda)\Omega^2+(\Gamma+\lambda)[(\Gamma+2\lambda)^2+4\nu^2]. \tag{7}$$

We have thus reduced the problem to finding the roots of two third-order polynomials $P_1(\lambda)=0$ and $P_2(\lambda)=0$.

A physical observable is the intensity of emitted light $\langle\mathcal{I}(t)\rangle$ which is equal to $\Gamma$ times the population in the excited state

$$\langle\mathcal{I}(t)\rangle\equiv\Gamma\left(\langle w\rangle_+ +\langle w\rangle_- +\frac{1}{2}\right)\equiv\Gamma\left(\langle w\rangle+\frac{1}{2}\right). \tag{8}$$

One can show that only the roots of $P_1(\lambda)=0$, denoted as $\{\lambda_1,\lambda_2,\lambda_3\}$, enter the solution of $\langle\mathcal{I}(t)\rangle$ [24]

$$\langle\mathcal{I}(t)\rangle=\mathcal{I}_{ss}+\sum_{n=1}^{3}\frac{e^{t\lambda_n}\Gamma(\lambda_n+\Gamma)((2\lambda_n+\Gamma)(2\lambda_n+4R+\Gamma)+4R\Gamma_{\rm SD})}{4R\Gamma\Gamma_{\rm SD}+(4R+\Gamma)(\Gamma^2+2\Omega^2)-8\lambda_n^2(\lambda_n+R+\Gamma)}, \tag{9}$$

where we have defined $\Gamma_{\rm SD}\equiv\nu^2/R$. The eigenvalue $\lambda=0$ [see Eq. (6)] yields the steady-state solution

$$\mathcal{I}_{ss}=\frac{\Gamma(4R+\Gamma)\Omega^2}{(4R+\Gamma)(\Gamma^2+2\Omega^2)+4R\Gamma\Gamma_{\rm SD}}. \tag{10}$$

This expression when $\Omega\to 0$ is the well-known Kubo-Anderson line shape at zero laser detuning.

We now focus our attention on the eigenvalues $\{\lambda_i\}$ to determine whether the solution is overdamped or underdamped. The motion is called overdamped if all eigenvalues $\{\lambda_i\}$ are real otherwise it is underdamped. The condition for overdamped behavior is that the discriminant $\mathcal{D}$ of $P_1(\lambda)$ be less than zero, explicitly, we have

$$\mathcal{D}=-16\,384\left[\left(\frac{\Gamma-\Gamma_{\rm SD}}{4}\right)^2-\Omega^2\right]R^4+512\{\Gamma^3-8\Omega^2(2\Gamma+5\Gamma_{\rm SD})+\Gamma_{\rm SD}[\Gamma^2+2\Gamma_{\rm SD}(\Gamma_{\rm SD}-2\Gamma)]\}R^3+64\{128\Omega^4+8(\Gamma^2+19\Gamma_{\rm SD}\Gamma+6\Gamma_{\rm SD}^2)\Omega^2-\Gamma^2[\Gamma^2+8(\Gamma-\Gamma_{\rm SD})\Gamma_{\rm SD}]\}R^2+64[\Gamma_{\rm SD}\Gamma^4+2\Omega^2(\Gamma-10\Gamma_{\rm SD})\Gamma^2+16\Omega^4(3\Gamma_{\rm SD}-2\Gamma)]R-1024\Omega^4\left[\left(\frac{\Gamma}{4}\right)^2-\Omega^2\right]<0. \tag{11}$$

Recall that for the noiseless case we have a single critical frequency $\Omega_c=\Gamma/4$. As shown in Fig. 1 in the presence of multiplicative noise the phase diagram of the motion is very rich:

(i) In the slow modulation $\nu\gg R$ strong coupling $\nu/\Gamma\gg 1$ regime, the solutions are always oscillatory and the critical frequency is 0 [indicated by 0 in Fig. 1(a)].

(ii) When $\nu\approx R$ we obtain overdamped motion when $\Omega_{C_1}<\Omega<\Omega_{C_2}$ so we have *two* critical frequencies [indicated by 2 in Fig. 1(a)].

(iii) In the fast modulation $R>\nu$ strong coupling $\nu/\Gamma>1$ limit, we find a single critical frequency similar to the noiseless case, except for a surprising line on which $\Omega_C=0$ [denoted with 0 in Fig. 1(a)].

(iv) In the weak noise limit $\nu\ll\Gamma$, the solution yields either two or three critical frequencies [see Fig. 1(b)]. Thus weak noise modifies the solution dramatically by doubling or tripling the number of critical frequencies of the system.

To understand better the phase diagram Fig. 1 we present in Fig. 2 the behavior of the solution of the optical Bloch equation in the *absence* of the multiplicative noise, i.e., $\nu=0$. The figure is a phase diagram in the Rabi frequency $\Omega/\Gamma$ and detuning $(\omega_L-\omega)/\Gamma$ plane showing the regions of over-

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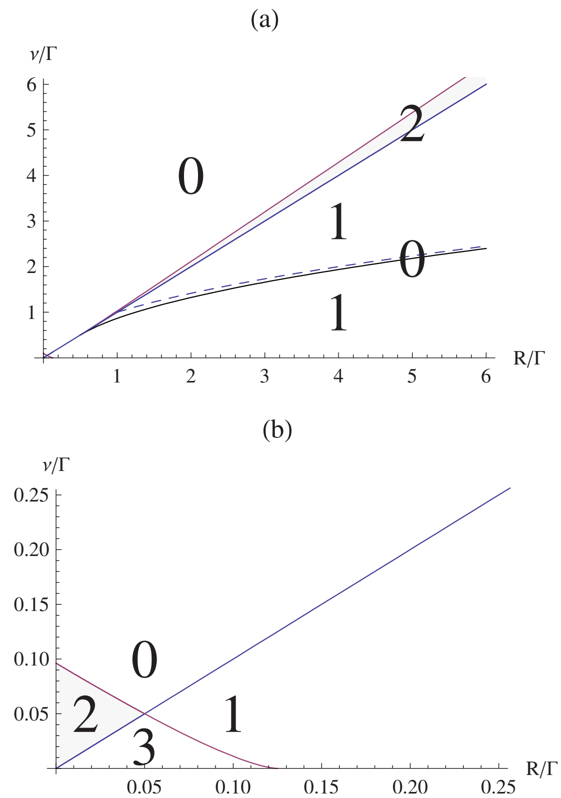


FIG. 1. (Color online) The phase diagram for the critical frequencies of the optical Bloch equation with multiplicative two state noise. The 0, 1, 2, and 3 indicate the number of critical frequencies as defined in the text. In (a) we see a line of zero critical frequency in the fast modulation limit $\nu<R$, which is well approximated by the cumulant expansion (the dashed black line $\nu/\Gamma=\sqrt{R/\Gamma}$). In the absence of spectral diffusion $\nu=0$ we have a single critical point, thus as shown in (b) the addition of weak noise $\nu/\Gamma\ll 1$ may strongly influence the critical frequency in the sense that we find there a phase with two or three critical frequencies.

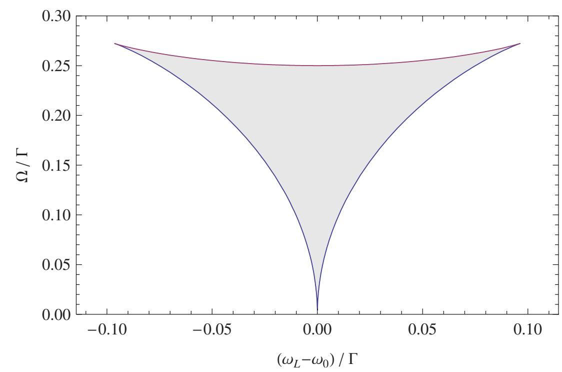


FIG. 2. (Color online) Phase diagram of the optical Bloch equation in the absence of the multiplicative noise. The darker area is the overdamped phase. For zero detuning $\omega_L-\omega_0=0$ the critical frequency is $\Omega_C=\Gamma/4$. Notice the cusp at zero detuning which makes the solutions unstable to multiplicative noise [25].

damped and underdamped behavior. Figure 2 illustrates that the noiseless system is unstable in the sense that for any small detuning and low enough Rabi frequency we get an oscillatory behavior while for zero detuning the solution is overdamped. This instability of the noiseless solution explains why even adding a weak perturbation $\nu\ll\Gamma$ strongly affects the system. Namely, for weak noise [Fig, 1(b)] we find either two or three critical frequencies instead of one for the noiseless case.

As mentioned before for slow modulation $\nu>R$ and strong coupling $\nu\gg\Gamma$ the motion is always underdamped. To understand this behavior we again refer to the noiseless case presented in Fig. 2, where we observe that large detuning means an oscillatory solution. Namely, oscillations for the noise free Bloch equation are induced by two mechanisms, the Rabi frequency and the detuning. Hence it is not surprising that strong and slow noise in Fig. 1(a) (i.e., $\nu>R,\Gamma$) may induce oscillations and the wipe out of the overdamped motion.

Far less trivial is the wipe out of overdamped motion in the fast modulation limit, i.e., the line of zero critical frequency in Fig. 1(a). To investigate this behavior we consider the limit $R\to\infty$. Then using Eq. (11) we find the critical frequency

$$\lim_{\nu,R\to\infty}\Omega_C=\left|\frac{\Gamma-\Gamma_{SD}}{4}\right|, \tag{12}$$

where the limit is taken with $\Gamma_{SD}$ remaining finite. We see that $\Omega_c=0$ when $\Gamma_{SD}=\Gamma$, namely, when $\nu/\Gamma=\sqrt{R/\Gamma}$. This line is shown in Fig. 1 as a dashed line.

Expanding the exact solution in $\Gamma_{SD}$, one can show that for $R>\Gamma/8$ any amount of noise will lead to a decrease in $\Omega_c$ according to

$$\Omega_C=\frac{\Gamma}{4}-\frac{2\Gamma_{SD}}{8-\Gamma/R}+O(\Gamma_{SD}^2), \tag{13}$$

where the leading $\Gamma/4$ term describes the noiseless case. The decrease in $\Omega_c$ is explained by the fact that the noise removes the system from zero detuning and hence solutions tend to be more oscillatory (i.e., the critical frequency is reduced). The surprising result is that by increasing the noise level we reach a limit where the critical frequency is zero. Such a behavior in the fast modulation limit could not be anticipated without our mathematical analysis. The behavior of $\Omega_c$ is illustrated in Fig. 3, which shows the decrease in the critical frequency until it reaches the value $\Omega_c=0$.

It is natural to ask if the behavior we found is general or limited to the example of a two state process. For this aim we have used the cumulant expansion [26] to investigate the critical frequency $\Omega_c$ of the system. We consider a stationary process $h(t)$ whose correlation function is $\langle h(t)h(t+\tau)\rangle=\exp(-R\tau)$. The cumulant expansion works well when the Kubo number $\nu/R$ is small. Within this approximation [26]

$$\frac{\partial\langle\vec{Z}\rangle}{\partial t}=(A_0+\nu^2K)\langle\vec{Z}\rangle, \tag{14}$$

where $A_0=A_0^{\pm}|_{\nu=0}$ and

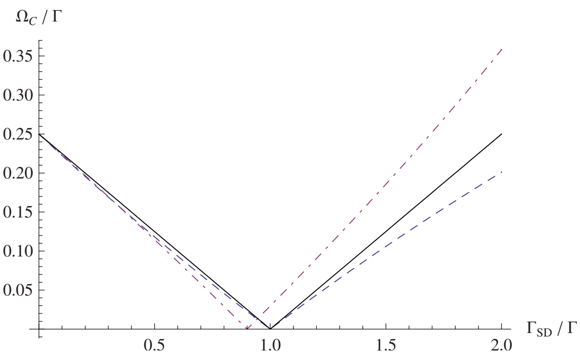


FIG. 3. (Color online) The critical frequency $\Omega_c$ as a function of $\Gamma_{SD}$ which is the measure of noise strength. For small noise levels, i.e., $\Gamma_{SD}/\Gamma<1$, the critical frequency $\Omega_c$ decreases as anticipated in Eq. (13). The figure illustrates the existence of $\Omega_c=0$ for a particular noise value. We show the critical frequency obtained from the exact solution (dot-dashed red line) and the cumulant approximation (dashed blue line) for $R=2.5\Gamma$. The solid line is the critical frequency $\Omega_c$ at $R\to\infty$ Eq. (12).

$$K=\int_0^\infty e^{-R\tau}A_1e^{A_0\tau}A_1e^{-A_0\tau}d\tau, \tag{15}$$

with $A_1=\begin{pmatrix}\sigma & 0\\ 0 & 0\end{pmatrix}$ and $\sigma=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$. Solving the integrals leads to cumbersome equations for $\Omega_c$. However in the limit $R\to\infty$ and $\nu\to\infty$ in such a way that $\Gamma_{SD}=\nu^2/R$ remains constant we find that Eq. (12) is valid and therefore the equation is not limited to the two state Kubo-Andersen model. This means that for a large class of stochastic processes, multiplicative noise induces zero critical frequency in the fast modulation limit and the curve $\nu/\Gamma=\sqrt{R/\Gamma}$, on which $\Omega_c=0$ shown in Fig. 1(a), is a general behavior.

To further validate the generality of our results we have solved semianalytically and with the help of Mathematica: (i) two state model with two nonidentical rates describing the transitions between up and down states and (ii) models with three states. These models show behaviors similar to our findings.

The two main effects found in this Rapid Communication: (a) noise inducing zero critical frequency and (b) the doubling or the tripling of the number of critical frequencies, *even for weak noise*, are found in linear multiplicative systems. The same effects cannot be found for linear systems driven by additive noise with zero mean since the averaged equations have the same critical frequency as the noiseless case. It would be interesting to investigate similar effects in nonlinear systems with additive or multiplicative noise.

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[1] A. Schenzle and H. Brand, Phys. Lett. A **69**, 313 (1979); Phys. Rev. A **20**, 1628 (1979).
[2] W. Horstenke and R. Lefever, *Noise-Induced Transition* (Springer-Verlag, Berlin, 1984).
[3] M. Gitterman, *The Noisy Oscillator: The First Hundred Years from Einstein Until Now* (World Scientific Publishing, Singapore, 2005), Chap. 10.
[4] K. Lindenberg and V. Seshadri, Physica A **109**, 483 (1981); V. Seshadri and K. Lindenberg, *ibid.* **115**, 501 (1982).
[5] C. Van den Broeck, J. M. R. Parrondo, and R. Toral, Phys. Rev. Lett. **73**, 3395 (1994).
[6] L. Gammaitoni, F. Marchesoni, E. Menichella-Saetta, and S. Santucci, Phys. Rev. E **49**, 4878 (1994).
[7] D. Sornette, Phys. Rev. E **57**, 4811 (1998).
[8] V. Berdichevsky and M. Gitterman, Phys. Rev. E **60**, 1494 (1999).
[9] M. Gitterman, Phys. Rev. E **69**, 041101 (2004).
[10] T. S. Biró and A. Jakovác, Phys. Rev. Lett. **94**, 132302 (2005).
[11] A. Manor and N. M. Shnerb, Phys. Rev. Lett. **103**, 030601 (2009).
[12] P. W. Anderson, J. Phys. Soc. Jpn. **9**, 316 (1954).
[13] In NMR but also for optical systems additional relaxations described by $T_1$ and $T_2$ may be included in the theory Eq. (1), however we will focus on the simplest case here.
[14] E. Barkai, Y. Jung, and R. Silbey, Annu. Rev. Phys. Chem. **55**, 457 (2004).
[15] Y. He and E. Barkai, Phys. Rev. Lett. **93**, 068302 (2004).
[16] H. Xu *et al.*, Phys. Rev. B **71**, 064512 (2005).
[17] Y. Tanimura, J. Phys. Soc. Jpn. **75**, 082001 (2006).
[18] R. Kubo, M. Toda, and N. Hashitsume, *Statistical Physics 2* (Springer, Berlin, 1995); *Statistical Physics 2* (Springer, New York, 1992).
[19] F. Sanda and S. Mukamel, Phys. Rev. Lett. **98**, 080603 (2007).
[20] The Bloch equations are identical to a system of two coupled damped harmonic oscillators $\ddot{u}+\frac{\Gamma}{2}\dot{u}+(\delta_L(t)^2+\frac{\Gamma^2}{4})u=\Omega\delta_L(t)w$ and $\ddot{w}+(\frac{3\Gamma}{2})\dot{w}+(\Omega^2+\frac{\Gamma^2}{2})w+(\frac{\Gamma^2}{4})=-\Omega\delta_L(t)u$. For a two state noise and zero detuning $\delta_L(t)^2=\nu^2$, hence for this case we have two simple harmonic oscillators with a stochastic coupling constant. Note that here we are interested only in $w(t)$.
[21] E. Geva and J. L. Skinner, J. Phys. Chem. B **101**, 8920 (1997).
[22] E. Barkai, A. V. Naumov, Yu. G. Vainer, M. Bauer, and L. Kador, Phys. Rev. Lett. **91**, 075502 (2003).
[23] A. I. Burshtein, Sov. Phys. JETP **22**, 937 (1966).
[24] In the noiseless limit of $\nu=0$, the eigenvalues of $P_1(\lambda_1)$ are $\lambda_1=-2R-\frac{\Gamma}{2}$ and $\lambda_{2,3}=-\frac{3\Gamma}{4}\pm\sqrt{\Gamma^2-16\Omega^2}$. The $R$-dependent eigenmode $\exp(\lambda_1 t)$ does not enter in the solution Eq. (9) since it is multiplied by zero and in the noiseless case our solution is independent of $R$ as it should. Note that the roots of $P_2(\lambda)=0$ enter in the solutions of $\langle u\rangle\equiv\langle u\rangle_+ + \langle u\rangle_-$ while $\langle v\rangle$ is determined by the roots of $P_1(\lambda)$.
[25] At zero detuning the eigenvalues of the matrix in Eq. (1) in the absence of noise are degenerate and real if $\Omega\ll\Gamma$. The addition of detuning removes the degeneracy of the eigenvalues and turns them to complex valued, as easily seen from Eq. (1) with small but finite detuning with $\Omega\simeq 0$ and in Fig. 2 near the unstable point $\Omega=0,\ \omega_L-\omega_0=0$.
[26] N. G. Van Kampen, *Stochastic Process in Physics and Chemistry* (North-Holland, Amsterdam, 1981).